PS8-Solution

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Question 1-Part a

The notion of *subgame* here is a bit tricky because when the seller makes a move, the buyer can be in a continuum of nodes based on the seller's type. So the buyer will play a *Nash Equilibrium* when he has to make a move and the seller seeing this, chooses a *Nash Equilibrium*. Since the game ends after the buyer makes a move, he cannot make any threads and change the Subgame perfect equilibrium and the seller knows this. So each play a NE and any equilibrium here would be SPNE.

Question 1-Part b

The buyer will not accept any $v > t_b$ because his payoff would be negative and he will accept anything $v \leq t_b$ since his payoff would be positive or zero. So he will weakly prefers values lower than his valuation. This behavior does not depend on the type of the seller[1] and the buyer follows this rule regardless of the realization of the seller's type.

[1] Of course v will depend on the seller's type, but this value is given from the standpoint of the buyer.

Question 1-Part c

The seller's expected Payoff would be:

$$\max_{\{v \geq t_s\}} \int_v^1 (v - t_s) dt_s = \max_{\{v \geq t_s\}} -v^2 + v(1 + t_s) - t_s$$

The F.O.C gives us the optimum value: $v^* = \dfrac{1 + t_s}{2}$ which satisfies the constraint.

Question 1-Part d

We have $0 \leq t_s \leq v \leq t_b \leq 1$ and the optimal v is $\frac{1+t_s}{2}$. So, $\frac{1+t_s}{2} \leq t_b \Rightarrow 0 \leq t_s \leq 2t_b - 1 \Rightarrow t_s \in [0, 2t_b - 1],\ t_b \in [\frac{1}{2}, 1]$.

- **i)**

$$Pr\{t_s < t_b\} = \int_{\frac{1}{2}}^{1} \Big[\int_{0}^{2t_b-1} dt_s\Big] dt_b = \frac{1}{2}$$

- **ii)**

$$Pr\{v^* < t_b\} = \int_{\frac{1+t_s}{2}}^{1} \Big[\int_{0}^{2t_b-1} dt_s\Big] dt_b = \frac{1}{4}$$

Question 1-Part e

$$\int_0^1 \left[\int_{t_s}^1 (t_b - t_s) dt_b \right] dt_s = \frac{1}{6}$$

Question 1-Part f

$$\int_0^1 \left[\int_{\frac{1+t_s}{2}}^1 (t_b - t_s) dt_b \right] dt_s = \frac{1}{8}$$

This shows a 25% drop in the efficiency.

Question 1-Part g

- **i)** Suppose the buyer lies and reports higher, then there is no gain for buyer. If he reports lower, then either trade does not occur in which case the buyer loses $\hat{t}_s$, or the trade does occur and the buyer does not gain. So truth telling is weakly dominant for the buyer. Similar argument applies to the seller. Since trade occurs whenever $\hat{t}_b > \hat{t}_s$, we have efficiency.

Question 1-Part g

- **ii)** Suppose there is another mechanism in which the third party pays strictly less and is efficient (trade occurs). We focus on truth-telling mechanisms. If the payment is strictly less, then someone should pay the difference ($\hat{t}_b - \hat{t}_s - c > 0$). Now there is incentive to lie, since the buyer will not get the seller's valuation and the seller does not get the buyer's valuation. In this case both will lie and go to the extreme: $\hat{t}_s = 0$, $\hat{t}_b = 1$ and there is no trade. So the third party should at least cover the difference in order to trade to occur.

Question 1-Part g

- **iii)**

$$\int_0^1 \int_0^{t_b} (t_b - t_s) dt_s dt_b = \frac{1}{6}$$

Question 2-Part a

Suppose that a_1 is never a best response for $\eta \in \Delta B$ and a_1 is not strictly dominated. Construct the following correspondence:

$$(\sigma, \eta) \longmapsto \{\sigma' | \sigma' \in \arg \max_{\{\sigma_1 \in \Delta A_1\}} E[U_1 | \sigma_1, \eta]\} \times$$
$$\{\eta' | E[U_1 | a_1, \eta'] \geq E[U_1 | \sigma, \eta']\}$$

The value of the correspondence at (σ, η) is the Cartesian product of two sets. The first part represents the set of best responses to η. It satisfies all the conditions of Kakutani's Fixed Point Theorem (KFPT) (i.e., it is upper-hemicontinuous, it is convex valued, and it is compact valued). The second part represents the set of all η' for which playing σ_1 is not better than playing a_1. Since a_1 is not strictly dominated by assumption, this second set is non-empty. It can be checked that the second part of the correspondence also satisfied the assumptions of KFPT.

Question 2-Part a

By KFPT, there is (σ'', η'') such that:

$$E[U_1|\sigma'', \eta''] \geq E[U_1|\sigma, \eta''], \ \forall\sigma \in \Delta A_1$$

$$E[U_1|a_1, \eta''] \geq E[U_1|\sigma'', \eta''], \ \ \forall\sigma \in \Delta A_1$$

Combining the two inequalities:

$$E[U_1|a_1, \eta''] \geq E[U_1|\sigma, \eta''], \ \ \forall\sigma \in \Delta A_1$$

This states that a_1 is a best response to η'' which contradicts our hypothesis.

Question 2-Part b

This is not true as illustrated in the example below.
Consider the following game, where $A_1 = \{U, D\}$, $A_2 = \{L, R\}$ and $A_3 = \{a, b, c, d\}$. The numbers are payoffs for player 3.

a

	L	R
U	9	0
D	0	0

b

	L	R
U	0	9
D	9	0

c

	L	R
U	0	0
D	0	9

d

	L	R
U	6	0
D	0	6

Question 2-Part b

Consider $\sigma_1(U, D) = (p, 1-p)$ and $\sigma(L, R) = (q, 1-q)$. Then:

$$u_3(\sigma_1, \sigma_2, a) = 9pq,$$

$$u_3(\sigma_1, \sigma_2, b) = 9p(1-q) + 9(1-p)q,$$

$$u_3(\sigma_1, \sigma_2, c) = 9(1-p)(1-q),$$

$$u_3(\sigma_1, \sigma_2, d) = 6pq + 6(1-p)(1-q).$$

If d is strictly dominated, then we can find $p, q \in [0, 1]$ such that:

$$9pq > 6pq + 6(1-p)(1-q), \; 9p(1-q) + 9(1-p)q > 6pq + 6(1-p)(1-q)$$

$$9(1-p)(1-q) > 6pq + 6(1-p)(1-q)$$

From the first equation, we have: $pq < 0$, which cannot happen as $p, q \in [0, 1]$. So d is never a best response.

Now let's show that d is not strictly dominated. Note that d can only be dominated by the mixed strategy of a and c. Suppose d is dominated by the strategy $\sigma_3(a, b, c, d) = (r, 0, 1-r, 0)$. If $s_{-i} = (U, L)$, then $9r > 6$ and if $s_{-i} = (D, R)$, then $9(1-r) > 6 \rightarrow 9r < 3$. So d is not strictly dominated.