PS6-Solution

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Question 1

Let $p, q, r \in \mathcal{L}$ and $q \succeq p \succeq r$. R.T.P. that there is an $\alpha \in [0, 1]$ such that $\alpha q + (1 - \alpha)r \sim p$.Define:
$U(p) = \sum_{i=1}^{n} p_i u(x_i) : \ q \succeq p \succeq r \Leftrightarrow U(q) \geq U(p) \geq U(r)$.If both inequalities are equalities, then any α will do the trick.Suppose on of the inequalities is strict, define:

$$\alpha = \frac{U(p) - U(r)}{U(q) - U(r)} \Rightarrow U(\alpha q + (1 - \alpha)r) = \alpha U(q) + (1 - \alpha)U(r)$$
$$= U(P)$$
$$\Rightarrow \alpha q + (1 - \alpha)r \sim p$$

Question 1

Independence: R.T.P that for all $p, q, r \in \mathcal{L}$ and every $\alpha \in [0,1] : q \succeq p \Leftrightarrow \alpha q + (1-\alpha)r \succeq \alpha p + (1-\alpha)r$.

$$\begin{aligned} U(\alpha q + (1-\alpha)r) = \sum (\alpha q_i + (1-\alpha)r_i)u(x_i) \geq \\ \sum (\alpha p_i + (1-\alpha)r_i)u(x_i) \\ = U(\alpha p + (1-\alpha)r) \end{aligned}$$

$\Rightarrow \alpha q + (1-\alpha)r \succeq \alpha p + (1-\alpha)r$

Question 1

"if" part is easy!For "only if" part, suppose
$p \succeq q \Leftrightarrow \sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{n} q_i u(x_i)$ and
$p \succeq q \Leftrightarrow \sum_{i=1}^{n} p_i v(x_i) \geq \sum_{i=1}^{n} q_i v(x_i)$ $\forall p, q \in \mathcal{L}$.
R.T.P. that there are $a, b \in \mathbb{R}$ such that $b > 0$ and
$v(x_i) = a + bu(x_i)$ $\forall i \in \{1, ..., n\}$.
According to class notes, there are best and worst elements of $\{e_1, ..., e_n\}$ and for all $p, q \in \mathcal{L}:\ e_n \succeq p \succeq e_1$. Since not all the elements of $\mathcal{L}$ are indifferent, we have: $e_n \succ e_1$.
Define a and b as the solutions to the following system of equations:

$$\begin{cases} v(x_n) = a + bu(x_n) \\ v(x_1) = a + bu(x_1) \end{cases}$$

R.T.P that $\forall e_i :\ v(x_i) = a + bu(x_i)$

Question 1

By mixture continuity, there exists $\alpha : \ \alpha e_n + (1-\alpha)e_1 \sim e_i$.

$$\begin{cases} U(\alpha e_n + (1-\alpha)e_1) = U(e_i) \\ V(\alpha e_n + (1-\alpha)e_1) = V(e_i) \end{cases} \Rightarrow \begin{cases} \alpha u(x_n) + (1-\alpha)u(x_1) = u(x_i) \\ \alpha v(x_n) + (1-\alpha)v(x_1) = v(x_i) \end{cases}$$

$$\Rightarrow \begin{cases} \alpha \overbrace{[a + bu(x_n)]}^{v(x_n)} + (1-\alpha) \overbrace{[a + bu(x_1)]}^{v(x_1)} = a + bu(x_i) \\ \alpha v(x_n) + (1-\alpha)v(x_1) = v(x_i) \end{cases}$$

$\Rightarrow v(x_i) = a + bu(x_i)$.

Question 2

R.T.P that for all $p, q, r \in \mathcal{L}$ and every $\alpha \in [0, 1] : q \succeq p \Leftrightarrow \alpha q + (1-\alpha)r \succeq \alpha p + (1-\alpha)r$.
"if part": Suppose $q \succeq p$. If $\alpha = 0$, we should have $r \succeq r$. Otherwise, there are two cases:

- $q_1 < p_1$: $\alpha q_1 + (1-\alpha)r_1 < \alpha p_1 + (1-\alpha)r_1 \Rightarrow \alpha q + (1-\alpha)r \succeq \alpha p + (1-\alpha)r$
- $q_1 = p_1$: $\alpha q_1 + (1-\alpha)r_1 = \alpha p_1 + (1-\alpha)r_1 \Rightarrow \sum_{i=2}^{n} q_i u(x_i) \geq \sum_{i=2}^{n} p_i u(x_i) \Rightarrow \sum_{i=2}^{n} (\alpha q_i + (1-\alpha)r_i)u(x_i) \geq \sum_{i=2}^{n} (\alpha p_i + (1-\alpha)r_i)u(x_i) \Rightarrow \alpha q + (1-\alpha)r \succeq \alpha p + (1-\alpha)r$

Question 2

"only if" part: Let $\alpha q + (1-\alpha) r \succeq \alpha p + (1-\alpha) r \ \forall \alpha \in [0,1]$. If $\alpha = 1$, then $q \succeq p$ and we are done!

- $\alpha q_1 + (1-\alpha) r_1 < \alpha p_1 + (1-\alpha) r_1 \Rightarrow q_1 < p_1 \Rightarrow q \succeq p$
- $\alpha q_1 + (1-\alpha) r_1 = \alpha p_1 + (1-\alpha) r_1 \Rightarrow q_1 = p_1$ and $\sum_{i=2}^{n} (\alpha q_i + (1-\alpha) r_i) u(x_i) \geq \sum_{i=2}^{n} (\alpha p_i + (1-\alpha) r_i) u(x_i) \Rightarrow \sum_{i=2}^{n} q_i u(x_i) \geq \sum_{i=2}^{n} p_i u(x_i) \Rightarrow q \succeq p$.

Question 2

Contraposition: Suppose there is an expected utility representation for $\succeq$ and everything holds except if $q_1 < p_1 \Rightarrow q \succ p$. Since no two elements of $\mathbb{X}$ are indifferent, we should have either $e_2 \succ e_3$ or $e_3 \succ e_2$. W.L.G suppose $e_3 \succ e_2$.If $e_2 \not\succ e_1$, then the above property does not hold and we are done. Now suppose $e_3 \succ e_2 \succ e_1 \Rightarrow u(x_3) > u(x_2) > u(x_1)$. Define

$$\alpha = \frac{u(x_2) - u(x_1)}{u(x_3) - u(x_1)} \Rightarrow u(x_2) = \alpha u(x_3) + (1-\alpha)u(x_1)$$

$$\Rightarrow e_2 \sim \alpha e_1 + (1-\alpha)e_3$$

which clearly violates "if $q_1 < p_1 \Rightarrow q \succ p$".

Question 2

When $n = 2$, there is an expected utility in the form of $u(x_1) = \gamma_1 < 0$ and $u(x_2) = \gamma_2 < 0$ with $|\gamma_1| > |\gamma_2|$.
This is because the condition $p_1 + p_2 = 1$ makes the preferences continuous.

Question 2

The independence axiom does not need mixture continuity, but we need it for utility representation. We have both in the case of $n = 2$, but not if $n \geq 3$.

Question 3

I skip the convexity proof. A good example is the following: There are two states of the world $\{0, 1\}$ and two actions $\{0, 1\}$. The utility over the actions and states is:

		State 0	State 1
p	0	1	0
$1-p$	1	0	1

$u(p, 1-p) = \max\{p, 1-p\}$

Question 3

Linearity: $a^*(p) \cap a^*(q) \neq \phi$.
"if part": Suppose a^{**} is the common element:

$$U(\alpha p + (1-\alpha)q) = \alpha \sum p_i u_i(a^{**}) + (1-\alpha) \sum q_i u_i(a^{**})$$
$$= \alpha U(p) + (1-\alpha)U(q)$$

"only if" part: Suppose $a^*(p) \cap a^*(q) = \phi$ and $a^*(\alpha)$ is the maximizer for $\alpha p + (1-\alpha)q$:

$$U(\alpha p + (1-\alpha)q) = \alpha \sum p_i u_i(a^*(\alpha)) + (1-\alpha) \sum q_i u_i(a^*(\alpha))$$
$$< \alpha U(p) + (1-\alpha)U(q)$$

which means $U(\cdot)$ is not linear.

Question 3

Here is an example why there might be a common element in the solution correspondence over lotteries and not all be the same. If the solution is the same over all lotteries, then they are all indifferent which is a special case for linearity.
Imagine a person who has to decide whether to work or not: $a \in \{0, 1\}$ where 0 means not working. The uncertainty is over the wages. Suppose wages are either 1 or 2: $w \in \{1, 2\}$. Also, suppose there exists the following lotteries:
$p = (\frac{1}{2}, \frac{1}{2}), q = (\frac{1}{3}, \frac{2}{3})$. The value of not working is 1.5. Clearly if the person is encountered with lottery p, she is indifferent between working or not and if she faces q, she chooses to work. The common solution is not working, i.e. $a = 0$.

Question 3

Strict convexity: $a^*(p) \cap a^*(q) = \phi$. The proof is similar to previous part so I skip it here.