PS6-Solution

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Question 1-Part a

- **Individual Rationality**: The allocation gives strictly higher utility to everyone relative to their endowment.
- **Pareto Optimality**: Suppose there exists an allocation that Pareto dominates this one, then if type b agent is strictly better off we have: $x_{1b} + x_{b2} > 8$. For type a agents, we have $x_{1a} + x_{2a} \geq 16$. Then aggregate feasibility is violated.

Question 1-Part b

It is not.Consider the following blocking:

- 1 agent of type a gets $(3.5, 3.5)$
- type b agent gets $(4.5, 4.5)$

Both gets higher utility (3.5 and 9 respectively) and it is feasible for their coalition: (8,8).

Question 1-Part c

Without person of type b, the most agents of type a can get is dividing their endowment which gives them utility of 2 while agent of type b is getting 8. So no coalition without type b can be in the core.
If agent of type b pairs up with an agent of type a, then only core allocation is $(4, 4)$ for type as and $(2, 6)$ for type b, otherwise the coalition of the previous part will block it.

Question 1-Part d

Using part b and c, consider the following: $x_a^i = (3, 3)$ and $x_b = (4, 8)$. This allocation is feasible and in the core since there is no blocking coalition. Also, the agent of type b gets 12 which is more than 8 $\Rightarrow x_b \succ_b w_b$.

Question 1-Part e

The utility function of the type as is not differentiable. If we have $p_1 < p_2$ or $p_2 < p_1$, then we have a corner solution for type b, but at this corner, type as are not maximizing. The only case is that the price of both goods being the same and we have one set of Walrasian equilibrium:

$$p = (1, 1),\ x_a^1 = x_a^2 = (2, 2),\ x_{1b} + x_{2b} = 8,\ x_{ib} \in [0, 8]$$

This is in the core since we have no blocking coalition.

Question 2

We proceed in several steps towards contradiction:

- **Step 1:** $|p \cdot (x - w_a)| = |\inf\{p \cdot (y - w_a) : y \succ_a x\}| = 0$
- **Step 2:** Suppose $x \notin D(p, a)$. Then there exists a feasible $x' \succ x$ and $p \cdot x' \leq p \cdot w_a$.
- **Step 3:** If $p \cdot x' < p \cdot w_a$, then construct x'' such that $p \cdot x'' = p \cdot w_a$ and $x'' \succ_a x' \Rightarrow x'' \succ_a x$.
- **Step 4:** By continuity, $\exists \alpha \in (0, 1) : \ \alpha x'' \succ_a x$.
- **Step 5:** $p \cdot (\alpha x'') = \alpha p \cdot x'' < p \cdot x'' = p \cdot w_a$.
- **Step 6:** $p \gg 0, \ x'' \in \mathcal{R}+^L \Rightarrow \inf\{p \cdot (y - w_a) : y \succ_a x\} < 0$.

Question 3-Part a

Since a social planner is maximizing, we have the following:

$$\max_{\{y_1,y_2,l_1,l_2\}} \ \log(y_1) + 2\log(y_2) + 2\log(24 - l_1) + \log(24 - l_2)$$

$$s.t \ y_1 + y_2 = 4\sqrt{l_1 + l_2}, \ l_1, l_2 \in [0, 24]$$

This give us the following allocations:

$$l_1 = \frac{40}{3}, \ l_2 = \frac{56}{3}, \ 1_1 = \frac{4}{3}\sqrt{32}, \ y_2 = \frac{8}{3}\sqrt{32}$$

Question 3-Part b

- **Firm:**

$$\max_{L} \ 4(p_1 + p_2)\sqrt{L} - wL \Rightarrow L = (\frac{2p}{w}), \ \pi = \frac{4p^2}{w}$$

- **Consumer 1:**

$$\max_{\{y_1, l_1\}} \ \log y_1 + 2\log(24 - l_1) \qquad s.t. \ \ p_1 y_1 = w l_1 + \pi_1$$

- Solve for y_1 from the budget constraint and substitute in the objective function, find l_1. Similarly for l_2. Use the fact that $L = l_1 + l_2$ and $y_1 = y_2 = y$ and normalize $w = 1$. This gives two equations in two unknowns p_1, p_2.
- **Equilibrium:**

$$l_1 = \frac{8}{3}, \ l_2 = \frac{40}{3}, \ w = 1, \ p_1 = \frac{2}{3}, \ p_2 = \frac{4}{3}, \ y_1 = y_2 = y = 16$$

Question 4-Part a

The initial endowment for everyone is (G, F). Setting $p_1 = 1$, individuals optimize:

- $\max \ V^i(x) = w(x_1) + w(x_1^2) + x_2^i \ \ s.t. \ x_1^i + px_2^i \leq G + pF$. The first order conditions here give us $w'(x_1^i) - \lambda \leq 0$ and $1 - p\lambda \leq 0$. For interior solution we have $w'(x_1^i) = \frac{1}{p}$. The strict concavity of $w(\cdot)$ implies that $x_1^1 = x_1^2$.
- $\max \ w(x_1^3) + x_2^3 \ \ s.t \ x_1^3 + px_2^3 \leq G + pF$. We have a similar first order conditions here that give us: $w'(x_1^3) = \frac{1}{p}$. So everyone consumes the amount G for the first good. From the budget constraints, we get that everyone consumes F of the second good. The price would be $\frac{1}{w'(G)}$.

Question 4-Part b

The social planner would maximize the sum of utilities:

$$\max\ 2w(x_1^1) + 2w(x_1^2) + w(x_1^3) + x_2^1 + x_2^2 + x_2^3$$

$$s.t. \quad x_1^1 + x_1^2 + x_1^3 = 3G,\ x_2^1 + x_2^2 + x_2^3 = 3F$$

This optimization gives us $x_2^1 = x_2^2 = x_2^3 = F$ and $x_1^1 = x_1^2 > x_1^3,\ 2x_1^1 + x_1^3 = 3G$. I got this allocation from the fact that $w(\cdot)$ is strictly concave and strictly increasing similar to previous part. The Walrasian Equilibrium that I found in the previous part is not in this set.