PS5-Solution

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Question 1-Part a

It is not consistent. First it is unrealistic. Second, with strictly monotone preferences, if $p \to 0$, then the demand set is empty which is not the case here.

Question 1-Part b

We cannot demand more than the total endowment, so $x_i \in [0, \sum_i w_i = w]$. Also, $p \in \Delta$. So for each $r \in [0,1]$, the set $E(r)$ is closed and bounded, hence it is compact.
Suppose the sequence $r^n \in [0,1]$ converges to some $r \in [0,1]$. If $(x^n, p^n) \in E(r^n)$ and $(x^n, p^n) \to (x, p)$, then I should show that $(x, p) \in E(r)$. The elements in $E(r)$ has a feature: they are all Walrasian Equilibrium. So total demand in the economy should be less than or equal to total endowment: $\sum_i x_i(p, r) \leq w$.

$$\sum_i^I x_i^n(p^n, r_i^n) \leq \sum_{i=1}^I w_i = w \Rightarrow \lim_{n\to\infty} \sum_i^I x_i^n(p^n, r_i^n) \leq \lim_{n\to\infty} w = w \Rightarrow$$

$$\sum_i^I \lim_{n\to\infty} x_i^n(p^n, r_i^n) \leq w \Rightarrow \sum_i^I x_i^n(p^n, r_i^n) \leq w \Rightarrow x(p, r) \in E(r)$$

So, $E(r)$ has a closed graph.

Question 1-Part c

We need to pick a sequence in $[0,1]$ than converges to 1 from below. Let $r_i^n = 1 - \frac{1}{n}\ \forall\ i$. Then for each n, we have a WE (x^n, p^n). Since $p^n \in [0,1]$, it is bounded and so it has a convergent subsequence. We call that p^{n_k} that converges to some $p \in \Delta$. The sequence $x^{n_k} = x^{n_k}(p^{n_k}, 1 - \frac{1}{n_k})$ converges to $x(p,1)$ which we know from part b that $x(p,1) \in E(r)$. So there is a WE with $r = (1, ..., 1)$.

Question 2

Assume that the set A has Lebesgue measure zero. Then for any $\epsilon > 0$, we have a countable collection of rectangles I_i such that $\sum_{i=1}^{\infty} vol(I_i) < \epsilon$ and their union covers A: $A \subseteq \cup_{i=1}^{\infty} I_i$. But A is nonempty, so there is $a \in A$. Also it is open: we have an open ball around a that lies in A:
$\exists\ r > 0 : B_r(a, r) = \{x : |x - a| < r\} \subset A \Rightarrow B_r(a, r) \subseteq \cup_{i=1}^{\infty} I_i$.
But there is a lower bound to the volume of $B_r(a, r)$:
$vol(B_r(a, r)) \geq \Pi_i^L |r - (-r)| = (2r)^L$. For $\epsilon < (2r)^L$, we cannot possibly have $B_r(a, r) \subseteq \cup_{i=1}^{\infty} I_i$. So A does not have Lebesgue measure zero.

Question 3

This is an application of the Transverlality theorem. First verify that there is sufficient differentiability: $f : \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are C^1 and $1 \geq 1 + max\{0, n - n\}$.
It suffices to prove that $F(x, w) = 0$ implies that $DF(x, w)$ has rank n. To that end we compute $DF(x, w) = D[f(x) + w]$.

Question 3

$$DF(x,w) = \begin{pmatrix} \frac{\partial(f_1+w_1)}{\partial x_1} & \cdots & \frac{\partial(f_1+w_1)}{\partial x_n} & \frac{\partial(f_1+w_1)}{\partial w_1} & \cdots & \frac{\partial(f_1+w_1)}{\partial w_n} \\ \frac{\partial(f_2+w_2)}{\partial x_1} & \cdots & \frac{\partial(f_2+w_2)}{\partial x_n} & \frac{\partial(f_2+w_2)}{\partial w_1} & \cdots & \frac{\partial(f_2+w_2)}{\partial w_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial(f_n+w_n)}{\partial x_1} & \cdots & \frac{\partial(f_n+w_n)}{\partial x_n} & \frac{\partial(f_n+w_n)}{\partial w_1} & \cdots & \frac{\partial(f_n+w_n)}{\partial w_n} \end{pmatrix}$$

$$DF(x,w) = \begin{pmatrix} \frac{\partial(f_1+w_1)}{\partial x_1} & \cdots & \frac{\partial(f_1+w_1)}{\partial x_n} & 1 & 0 & 0 & \cdots & 0 \\ \frac{\partial(f_2+w_2)}{\partial x_1} & \cdots & \frac{\partial(f_2+w_2)}{\partial x_n} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \cdots & 1 & 0 \\ \frac{\partial(f_n+w_n)}{\partial x_1} & \cdots & \frac{\partial(f_n+w_n)}{\partial x_n} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Note that DF has rank n because the last n columns are linearly independent.

Question 3

Then by the Transversality Theorem there exists a Lebesgue-measure zero set Ω_0, $\Omega_0 \subset \mathbb{R}^n$ such that

$$\omega \notin \Omega_0 \implies \{F(x, \omega) = 0 \implies D_x F(x, \omega) \text{ has rank } n\}$$

or equivalently

$$\omega \notin \Omega_0 \implies \{f(x) + \omega = 0 \implies D_x f(x) \text{ has rank } n\}$$

The Transversality Theorem concludes also (because the dimension $n = m$) that there is a local, C^1 function $x^*(\omega)$ characterized by $F(x^*(\omega), \omega) = 0$ or equivalently $f(x^*(\omega)) = -\omega$.
The function $x^*(\omega)$ is the function h in the statement of the problem.

Question 3

Finally, one needs to show that the measure-zero set with the desired property is closed.
Let $\Omega_1 = \{\omega \in \mathbb{R}^n : \exists x \in \mathbb{R}^n,\ F(x,\omega) = 0 \wedge \det D_x F(x,\omega) = 0\}$.
Let $\omega^n \in \Omega_1$, $\omega^n \to \bar{\omega}$. Let x^n such that
$F(x^n, \omega^n) = 0 = f(x^n) + \omega^n$. Since $\omega^n \to \bar{\omega}$, $f(x^n) \to -\bar{\omega}$. In a subsequence $x^n \to \bar{x}$ and it must be that $f(\bar{x}) = -\bar{\omega}$. Then
$0 = \det D_x F(x^n, \omega^n)$ and $\det D_x F(x^n, \omega^n) \to \det D_x F(\bar{x}, \bar{\omega})$.
Since Ω_1 has measure zero, the exercise is complete.