PS3-Solution

Mehrdad Esfahani

Arizona State University

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Question 1-part a

I proceed by analogous steps in the proof of Second Welfare Theorem in MWG
Step 1: Assume (x^*, y^*) is a Pareto Optimal allocation. Define $V = \{x \in X : x \succ x^*\}$. V is a convex set because our utility is quasiconcave. I want to show that $V \cap \hat{Y} = \Phi$. Suppose it is not the case. Then we have $x \in V \; and \; x \in \hat{Y}$. So it is feasible and $x \succ x^*$. So x^* is not PO which is a contradiction.

Question 1-part a

Step 2: We have two convex and nonempty sets from the previous step that has an empty intersection. So by Separating Hyperplane Theorem, we have a $p^* \in \mathbb{R}^L$ $p^* \neq 0$ such that $\inf p^* V \geq \sup p^* \hat{Y}$. If something is in the closure of the set V which we denote by $\hat{V}$, then for some $x^n \to x$ and since we have the property for every x^n, and the dot product is continuous, then we have $p^* x \geq \sup p^* \hat{Y}$.

Question 1-part a

Step 3: We have strong monotonicity here for preference relation. If $x^* \notin \bar{V}$, then we have a bundle very close to it which is also in $\hat{Y}$ such that it is strictly preferred to x^*. This means the new bundle is in $V \cap \hat{Y}$, which contradicts step 1. So, $x^* \in \bar{V} \cap \hat{Y}$. This means: $x^* = y^* + w$.

Question 1-part a

Step 4: From step 2, we have $p^*(w + y^*) \geq p^* y \forall\ y \in \hat{Y}$. This means that $p^* y^* \geq p^* y\ \forall\ y \in Y$. This gives us firm optimization.

Question 1-part a

Step 5: If we have $x \in V$, then since $x^* \in \hat{Y}$, we have from step 2 that $p^*x \geq p^*x^*$. So, we do not yet have preference maximization. We need a strict inequality. I try to come up with this in the next step.

Question 1-part a

Step 6: First we have to show that $p^* \gg 0$. Form step 4, we know that $p^* y^* \geq 0$. Then if some element of the price vector is negative, free disposal allows me to get rid of some of that production and come up with y' such that: Everything in y' is the same as y^* except for the negative element which is $y'_i = y^*_i - \epsilon \; \epsilon > 0$ and so: $p^* y' > p^* y$. This contradicts step 4. So, price is strictly positive. If the inequality in step 5 is not strict, then we have $x \succ x^*$ and $p^* x^* = p^* x$. If I choose x' that equals to x except for the *ith* element which is strictly less than the *ith* element in x, then by continuity of preference relation we know that as this difference goes to zero, we have $x' \succ x$. we also know that $p^* x' < p^* x = p^* x^*$. This would contradict step 5. So, we showed that (x^*, y^*) is a WE.

Question 1-Part b

We just need to show there exists a PO allocation. Since we want to maximize a continuous utility function over a compact set $(\hat{Y})$[1], we have a solution due to Weierstrass.

[1] This set is closed and bounded by assumption, hence compact.

Question 2-Part a

Since the price vector is not zero by assumption, there should be at least one good with positive price: $p_i > 0$. Also, the aggregate endowment vector strictly positive, so there should be someone (j) with positive endowment of good i. So $p.w_j \geq p_i \cdot w_{ij} > 0$.

Question 2-Part b

We assumed that $\parallel z(p^n) \parallel_\infty \nrightarrow \infty$. So, if $\parallel z_j(p^n) \parallel_\infty \rightarrow \infty$, then $\parallel z(p^n) \parallel_\infty \rightarrow \infty$. Therefore, $\parallel z_j(p^n) \parallel_\infty \nrightarrow \infty$.

Question 2-Part c

From the previous part, $z_j(p^n)$ is bounded above. From the assumption, it is also bounded below. So by Bolzano-Weierstrass theorem, this bounded sequence has a convergent subsequence and the point of convergence stays in the space of the sequence itself which is $\mathbb{R}^L$.

Question 2-Part d

WLOG, assume $p_1 = 0$ and $p_2 > 0$. Construct $y \in \mathcal{R}^L$ such that $y = (n, -\delta, 0, ..., 0)$, for some $n >> 0$ and $\delta > 0$. Take $\delta \to 0$. By strong monotonicity and continuity, $x + y + \omega_j \succ x + \omega_j$. Since $p_1 = 0$ and $p_2 > 0$, $p \cdot (x + y + \omega_j) < p \cdot (x + \omega_j)$.

Question 2-Part e

By (c), we know $\forall \epsilon > 0,\ \exists K \in N\ s.t.\ \forall k \in K,\ d(z_j(p^{n_k}), x) < \epsilon$.
Pick some $\epsilon < \delta$, then
$\exists K \in N\ s.t.\ \forall k \in K,\ p \cdot (x + y + \omega_j) < p \cdot (z_j(p^{n_k} + \omega_j)$. Also by continuity, we have $x + y + \omega_j \succ z_j(p^{n_k}) + \omega_j$.

Question 2-Part f

Since $p \cdot \omega > 0$, so we can always find some y, such that $\exists K \in N \ s.t. \ \forall k \in K, \ p \cdot (x + y + \omega_j) < p \cdot (z_j(p^{n_k} + \omega_j)$ and $x + y + \omega_j \succ z_j(p^{n_k}) + \omega_j$. This means increasing a positive amount of the good with zero price, j gets better off. This contradicts with $z_j(p^{n_k}) \to x$.

Question 3-Part a

The function is a finite sum of continuous functions in the defined domain, so it is continuous. For positive values for $x_i,\ i = 1, 2, 3$, the first partial derivatives of the function is strictly positive. So we have strong monotonicity. Also, each part of the function is strictly concave in x_i, So:

$$U(\lambda x_1 + (1-\lambda)x_1', \lambda x_2 + (1-\lambda)x_2', \lambda x_3 + (1-\lambda)x_3') =$$
$$\sqrt{\lambda x_1 + (1-\lambda)x_1'} + \sqrt{\lambda x_2 + (1-\lambda)x_2'} + \lambda x_2 + (1-\lambda)x_2' +$$
$$\frac{\lambda x_3 + (1-\lambda)x_3'}{1 + \lambda x_3 + (1-\lambda)x_3'} >$$
$$\lambda U(x_1, x_2, x_3) + (1-\lambda)U(x_1', x_2', x_3')$$

So we have strict concavity.

Question 3-Part b

$$U(x_1, x_2 + x_3, 0) = \sqrt{x_1} + \sqrt{x_2 + x_3} + x_2 + x_3 \Rightarrow$$
$$U(x_1, x_2 + x_3, 0) - U(x_1, x_2, x_3) =$$
$$\sqrt{x_2 + x_3} - \sqrt{x_2 + x_3} + \frac{x_3^2}{1 + x_3}.$$

Since $x_3 > 0$, we have $\sqrt{x_2 + x_3} > \sqrt{x_2} + x_3$. So, $U(x_1, x_2 + x_3, 0) > U(x_1, x_2, x_3)$.

Question 3-Part c

In this case, the budget set remains the same:
$p_1x_1+p_2(x_2+x_3)+p_2\cdot 0 = p_1x_1+p_2x_2+p_2x_3 = p_1x_1+p_2x_2+p_3x_3$.
From Part b, we know that $U(x_1, x_2 + x_3, 0) > U(x_1, x_2, x_3)$. So we are better of with no x_3.

Question 3-Part d

The price of the second and the third good are the same. So from Part c, we have $x_3(p^n) = 0$. The utility function tells us that the demand correspondence is actually a continuous function. So, as $n \to \infty$, we have $x_3(p^n) = 0$.

Question 3-Part e

With $x_3 = 0$, we want to maximize $\sqrt{x_1} + \sqrt{x_2} + x_2$ subject to $(1 - \frac{2}{n})x_1 + \dfrac{x_2}{n} = p \cdot w = 1$. We are allowed to take the first order conditions:

$$\frac{1}{1 + 2\sqrt{x_2}} \sqrt{\frac{x_2}{n - x_2}} = \sqrt{n - 2} \qquad (1)$$

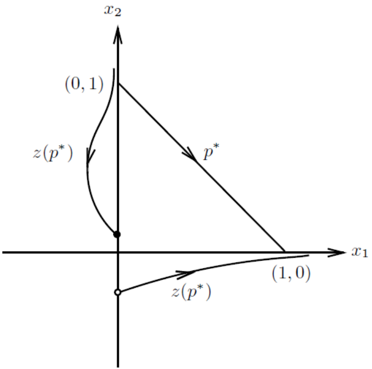
If $n \to \infty$, the RHS goes to infinity while with a bounded x_2, the LHS goes to zero and violate the equation. So, $x_2(p^n) \to \infty$.

Question 3-Part f

The boundary condition should hold because of strict monotonicity of the preference relation. So, when some price goes to zero, the demand of some good should go to infinity (x_2 here). Because if everything remains finite, then we can get more utility by increasing the demand of the good with zero price. We can also have finite demand for some good (x_3 here) with its price being zero. So, the statement is true and boundary condition is not violated.

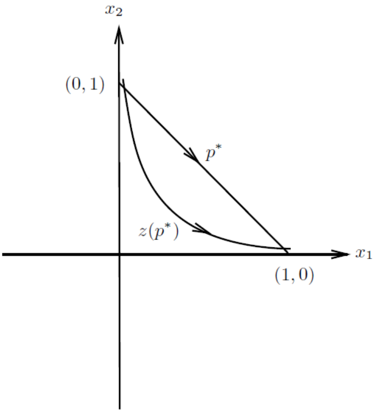
Question 4-Part a

Violation of continuity



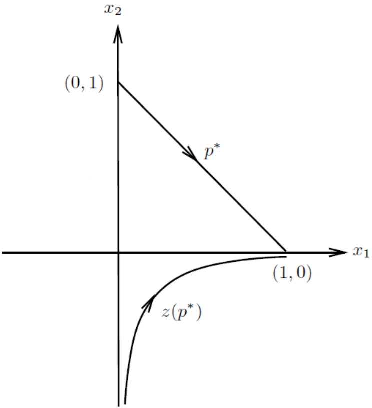
Question 4-Part b

Violation of Walras' Law



Question 4-Part c

Violation of boundedness below



Question 4-Part d

Violation of the boundary condition