PS2-Solution

Mehrdad Esfahani

Arizona State University

Fall 2016

Question 1

Contraposition: suppose $x_1, x_2 \in d(p,w)$ and $x_1 \neq x_2$. We need to show that $\succeq$ is not strictly convex.

$$x_1 \in d(p,w) \Rightarrow p \cdot x_1 \leq w$$

$$x_2 \in d(p,w) \Rightarrow p \cdot x_2 \leq w$$

This means that $p \cdot (\frac{1}{2}x_1 + \frac{1}{2}x_2) \leq w \Rightarrow \frac{1}{2}x_1 + \frac{1}{2}x_2 \in B(p,w)$. From the demand set we know that

$$x_1 \succeq x_2 \quad x_2 \succeq \frac{1}{2}x_1 + \frac{1}{2}x_2$$

Transitivity of $\succeq$ gives us $x_1 \succeq \frac{1}{2}x_1 + \frac{1}{2}x_2$ which means that $\frac{1}{2}x_1 + \frac{1}{2}x_2 \not\succ x_1$. Therefore, $\succeq$ is not strictly convex.

Question 1

Let $x_1, x_2 \in d(p,w)$ and $\lambda \in [0,1]$. We need to show that $\lambda x_1 + (1-\lambda)x_2 \in d(p,w)$.

$$x_1 \in d(p,w) \Rightarrow p \cdot x_1 \leq w \Rightarrow p \cdot \lambda x_1 \leq \lambda w$$

$$x_2 \in d(p,w) \Rightarrow p \cdot x_2 \leq w \Rightarrow p \cdot (1-\lambda)x_2) \leq \lambda w$$

$\Rightarrow p \cdot (\lambda x_1 + (1-\lambda)x_2) \leq w \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in B(p,w)$. Let $y \in B(p,w)$ be an arbitrary element in $B(p,w)$. From the demand sets and convexity of $\succeq$ we have:

$$x_1 \succeq x_2 \quad \lambda x_1 + (1-\lambda)x_2 \succeq x_2 \succeq y$$

Transitivity gives us
$\lambda x_1 + (1-\lambda)x_2 \succeq y \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in d(p,w)$.

Question 2

For any $\lambda \in R_{++}$ and $x \in d(p, w)$, straightforward algebra reveals that $\lambda x \in B(p, \lambda w)$. We need to show that $\lambda x \in d(p, \lambda w)$.

$\forall y \in B(p, \lambda w): \; p \cdot y \leq \lambda w \Rightarrow p \cdot (\frac{1}{\lambda} y) \leq w \xRightarrow{x \in d(p,w)} u(\frac{1}{\lambda} y) \leq u(x) \xRightarrow{u(\cdot) is HD1} \frac{1}{\lambda} u(y) \leq u(x) \Rightarrow u(y) \leq u(\lambda x) \quad \square$

This means that $v(p, \lambda w) = u(\lambda x) = \lambda u(x)$. Since $x \in d(p, w)$ we have $u(x) = v(p, w)$. Therefore, $v(p, \lambda w) = \lambda v(p, w)$. Wealth expansion paths are straight lines from the origin.

Question 2

Strict quasiconcavity implies that the demand correspondence is in fact a function. Here we need to assume differentiability:

$$d(p,\lambda w) = \lambda d(p,w) \xrightarrow{\frac{\partial}{\partial \lambda}} \sum_{i=1}^{L} \frac{\partial d}{\partial p_i} \cdot \frac{\partial p_i}{\partial \lambda} + \frac{\partial d}{\partial \lambda w} \cdot \frac{\partial \lambda w}{\partial \lambda} = d(p,w)$$

Setting $\lambda = 1$, we have $\dfrac{\partial d}{\partial w} \cdot w = d(p,w) \Rightarrow \dfrac{\partial d}{\partial w} \cdot \dfrac{w}{d} = 1.$

Question 3

We know that a strictly increasing transformation does not change the utility representation. Consider this one: $x^{\frac{1}{a+b}}$. This means that the assumption $a+b=1$ is without loss of generality.

Question 3

Ode to Domination

If we divide wealth equally over the two goods, we are strictly better if one good is zero. So we can get rid of the boundary of the budget set where at least one good has zero quantity.
This means that $x_i = 0$ for some i is *strictly dominated* by $(\frac{w}{2(p_1 + p_2)}, \frac{w}{2(p_1 + p_2)})$.
This allows us to use a monotone transformation such as logarithm

Question 3

The only other trick here is that since the utility function is strictly increasing in both arguments, the budget constraint binds. So the demands for goods are

$$x_1(p, w) = a \cdot \frac{w}{p_1}, \quad s_1 = a$$

$$x_2(p, w) = b \cdot \frac{w}{p_2}, \quad s_2 = b$$

The budget share is independent of prices and wealth.

Part c

Let $(x_1, x_2),\ (x_1', x_2') \in R_+^2$ such that $(x_1, x_2) \sim (x_1', x_2')$. Let $\lambda \in R_+$. We need to show that $(\lambda x_1, \lambda x_2) \sim (\lambda x_1', \lambda x_2')$.
Easy algebra using the utility function gives us the result.
Also it is easy to show based on the demand functions that the own price elasticity is 1, the cross price is zero and the wealth is -1.

Question 3

If a monotone transformation of a function is quasiconvex, then the function itself is quasiconvex. Lets look at $\log v(p,w)$.The hessian matrix is

$$H = \begin{bmatrix} \frac{a}{p_1^2} & 0 \\ 0 & \frac{b}{p_2^2} \end{bmatrix}$$

All the eigenvalues of this matrix are strictly positive which means that $\log v(p,w)$ is strictly convex and quasiconvex. Simple algebra gives us the rest of the results.

Question 4

It is not optimal to choose $x_i = x_i^0$ for some i. This means we can take logs following the domination argument in question 3.

$$\max_{x_i} \sum_{i=1}^{L} \alpha_i \log(x_i - x_i^0)$$

$$s.t. \quad p \cdot x \leq w \;\text{ and }\; x \gg x^0$$

We know the budget constraint binds so we can form the Lagrangian:

$$\mathcal{L} = \sum_{i=1}^{L} \alpha_i \log(x_i - x_i^0) - \lambda(p \cdot x - w)$$

Question 4

F.O.C: $\dfrac{\alpha_i}{x_i - x_i^0} = \lambda p_i \Rightarrow \alpha_i = \lambda(x_i - x_i^0)p_i$

We sum over all i to find the multiplier: $\lambda = \dfrac{1}{w - p \cdot x^0}$. This gives us x_i and the claim follows.

Question 4

$s_i = \frac{p_i x_i}{w} = \alpha_i(1 - \frac{p \cdot x^0}{w}) + \frac{p_i x_i^0}{w}$ From this, it is straightforward to show that own price elasticity is positive, cross price is negative and wealth is undetermined.

To show that it is nonhomothetic, look at the following example: $x^0 \neq 0$, so without loss of generality, assume that $x_1^0 \geq 0$. Consider two bundles:

$x_1 = (x_1^0, ..., x_L^0)$ and

$x_2 = (x_1^0, x_2^0 + 1, ..., x_L^0 + 1) \Rightarrow u(x_1) = u(x_2) \Rightarrow x_1 \sim x_2$.

$u(2x_1) = (x_1^0)^{\alpha_1} \prod_{i=2}^{L} (x_i^0)^{\alpha_i}$

$u(2x_2) = (x_1^0)^{\alpha_1} \prod_{i=2}^{L} (x_i^0 + 2)^{\alpha_i} \Rightarrow u(2x_2) > u(2x_1)$. This means that $2x_1 \nsim 2x_2$

Question 5

Define $e = (1, 0, ..., 0)$. I need to show that
$d(p, w + \alpha) = d(p, w) + \alpha \cdot e$.
$x \in B(p, w) \Leftrightarrow x + \alpha \cdot e \in B(p, w + \alpha)$.
First Step: $x \in d(p, w) \Rightarrow x + \alpha \cdot e \in d(p, w + \alpha)$
For any $y \in B(p, w + \alpha)$, we have $y - \alpha \cdot e \in B(p, w) \xRightarrow{x \in d(p,w)}$
$x \succeq y - \alpha \cdot e \xRightarrow{quasilinearity} x + \alpha \cdot e \succeq y \Rightarrow x + \alpha \cdot e \in d(p, w + \alpha)$
Second Step: $x + \alpha \cdot e \in d(p, w + \alpha) \Rightarrow x \in d(p, w)$
For any $y \in B(p, w) \Rightarrow y + \alpha \cdot e \in B(p, w + \alpha) \xRightarrow{x + \alpha \cdot e \in d(p, w + \alpha)}$
$x + \alpha \cdot e \succeq y + \alpha \cdot e \xRightarrow{quasilinearity} x \succeq y \Rightarrow x \in d(p, w) \quad \square$
$$\frac{\partial d_1(p, w)}{\partial w} = 1$$

Question 5

We know that we can represent this relation with a function of the form $u(x) = x_1 + \phi(x_2, ..., x_L)$.
We showed that $d(p, w + \alpha) = d(p, w) + \alpha \cdot e \Rightarrow d(p, w) = d(p, 0) + w \cdot e \Rightarrow v(p, w) = w + \phi(d(p, 0)) = w + \psi(p)$

Question 5

If $p_2 x_2(p, 0) \geq w$, then the nonnegativity constraint binds and the demand is $(0, \frac{w}{p_2})$.
If $p_2 x_2(p, 0) < w$, then the constraint is irrelevant and the demand for good 2 does not respond to w.

Question 5

$$\max \ x_1 - \frac{1}{x_2}$$

$$x_1 + px_2 \leq w \quad x_1, x_2 \geq 0$$

Note that if $x_2 \to 0 \Rightarrow u \to -\infty$. So the constraint for x_2 never binds. If the constraint for x_1 does not bind, then the budget constraint binds and we can write $x_1 = w - px_2$. Replacing this into the utility function and maximizing gives us the following demand function:

$$d(p,w) = \begin{cases} (w - \sqrt{p}, \frac{1}{\sqrt{p}}) & \text{if } w - \sqrt{p} \geq 0; \\ (0, \frac{w}{p}) & \text{if } w - \sqrt{p} < 0. \end{cases}$$

Question 5

The preferences are not homothetic.
Counterexample: $u(1,1) = u(2,0.5) \Rightarrow (1,1) \sim (2,0.5)$
Choose $\lambda = 2$:
$u(2,2) = 1.5,\ u(4,1) = 3 \Rightarrow (2,2) \nsim (4,1)$