# PS1-Solution

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# Question 1-Part i

Irreflexitivity: By completeness:

$$\forall x \in X \to x \succeq x \to x \nsucc x \Rightarrow \succ \text{ is irreflexive.}$$

Transitivity: Suppose

$$x \nsucc z \text{ and } x \succ y$$

Since $\succeq$ is complete, we have $z \succeq x$ and $x \succeq y$. From transitivity of $\succeq \Rightarrow\ z \succeq y \Rightarrow y \nsucc z$ □

# Question 1-Part ii

By completeness:

$$\forall x \in X \to x \succeq x \Rightarrow x \sim x.$$

Transitivity:

$$x \sim y \Rightarrow x \succeq y$$

$$y \sim z \Rightarrow y \succeq z$$

From transitivity of $\succeq \Rightarrow x \succeq z$.

$$y \sim z \Rightarrow z \succeq y$$

$$x \sim y \Rightarrow y \succeq x$$

From transitivity of $\succeq \Rightarrow z \succeq x \Rightarrow x \sim z$.
Symmetry: $x \sim y \Rightarrow x \succeq y$ and $y \succeq x \Rightarrow y \sim x$. □

# Question 1-Part iii

By contraposition: Suppose $x \nsucc z$ and $y \succeq z$. we have to prove $x \nsucc y$.
Since $\succeq$ is complete, $x \nsucc z \Rightarrow z \succeq x$. So, we have $y \succeq z \succeq x$. By transitivity of $\succeq$, we get $y \succeq x \Rightarrow x \nsucc y$. □

# Question 2

Suppose $x, y \in X$. Since $u(\cdot)$ represents $\succeq$, we have:

$$x \succeq y \iff u(x) \geq u(y)$$

We know that $f(\cdot)$ is strictly increasing, so

$$f(u(x)) \geq f(u(y)) \Rightarrow v(x) \geq v(y)$$

This means that $v(\cdot)$ represents $\succeq$ as well. $\square$

# Question 3

**Example**

1. Place:One acre of land in Tempe and one acre in Phoenix.
2. State:Getting 1$ if the coin turns up head and nothing when it turns up tail.

# Question 4-Part a

"if part":
let $x, y \in X$ and $x \sim y$ and $\lambda \in \mathcal{R}_+$. We need to show: $\lambda x \sim \lambda y$.

$x \sim y \Rightarrow u(x) = u(y) \Rightarrow \lambda u(x) = \lambda u(y) \xrightarrow{u(\cdot):\ HD1} u(\lambda x) = u(\lambda y) \Rightarrow \lambda x \sim \lambda u.$ □

"only if" part:

We have to show that the utility function is homogeneous of degree 1 if the underlying preference relation is *homothetic*. Let's use the same line of reasoning as it was explained in the lectures.

$$\forall\ x \in X,\ \ u(x)e \sim x \xrightarrow[\lambda \geq 0]{\succeq:\ homothetic} \lambda u(x)e \sim \lambda x$$

By construction, we know that $\lambda x \sim u(\lambda x)e$[1].Hence, by transitivity of $\sim$ we have $\lambda u(x)e \sim u(\lambda x)e$.Using the definition of $u(\cdot)$, we have $u(\lambda x) = \lambda u(x)$. □

**Remark**

You can use the result of this question later in the course without proof.

[1] I replaced $x$ with $\lambda x$.

# Question 4-Part b

## Lemma

Let $u(x)$ be $C^1$ and homogeneous of dgree 1, then partial derivatives of $u(\cdot)$ are homogeneous of degree 0.
Proof:

$$u_i(\lambda x_1, ..., \lambda x_n) = \lim_{h\to 0} \frac{u(\lambda x_1, ..., \lambda x_i + h, ...) - u(\lambda x_1, ..., \lambda x_i, ...)}{h}$$

$$= \lim_{h\to 0} \frac{u(x_1, ..., x_i + \frac{h}{\lambda}, ..., x_n) - u(x_1, ..., x_n)}{\frac{h}{\lambda}} = u_i(x_1, ..., x_n)$$

$MRS_{ij}(\lambda x) = \dfrac{u_i(\lambda x)}{u_j(\lambda x)} = \dfrac{u_i(x)}{u_j(x)} = MRS_{ij}(x)$.So the wealth expansion paths are straight lines that pass through the origin.

# Question 5

"if part":
Suppose $u(y,m) = \Phi(y) + m$:

i. let $y \in Y$ and $m, m' \in \mathbb{R}$:
$(y,m') \succeq (y,m) \iff \Phi(y) + m' \geq \Phi(y) + m \iff m' \geq m$.

ii. let $y, y' \in Y$ and $m, m' \in \mathcal{R}$ and $\delta \in \mathcal{R}$:
$(y',m') \succeq (y,m) \iff \Phi(y') + m' \geq \Phi(y) + m \iff$
$\Phi(y') + m' + \delta \geq \Phi(y) + m + \delta \iff (y', m' + \delta) \succeq (y, m + \delta)$

iii. Let $y, y' \in Y$. Let $m = 0$ and $m' = \Phi(y) - \Phi(y')$. We have
$\Phi(y) + m = \Phi(y') + m' \Rightarrow (y,m) \sim (y',m')$.

$\square$

# Question 5

"only if" part:

Fix $y_0 \in Y$ and consider $(y, m) \in X$. By (iii) there are real numbers $m_0$ and $m_1$ such that $(y_0, m_0) \sim (y, m_1)$. By (ii): $(y_0, m_0 + m - m_1) \sim (y, m_1 + m - m_1) = (y, m) \Rightarrow (y, m) \sim (y_0, m + m_0 - m_1)$.

Since I fixed $y_0$ and chose $y$ arbitrarily, then $m_0 - m_1$ only depends on $y$. Lets call it $\Phi(y)$. Hence for every $y \in Y$ and $m \in \mathcal{R}$: $(y, m) \sim (y_0, \Phi(y) + m)$.

# Question 5

Now consider two consumption sets: $(y_1, m_1)$ and $(y_2, m_2)$:
$(y_1, m_1) \sim (y_0, \Phi(y_1) + m_1)$
$(y_2, m_2) \sim (y_0, \Phi(y_2) + m_2)$

Suppose[2]:

$$(y_1, m_1) \succeq (y_2, m_2) \leftrightarrow (y_0, \Phi(y_1) + m_1) \succeq (y_0, \Phi(y_2) + m_2) \overset{(i)}{\longleftrightarrow} \Phi(y_1) + m_1 \geq \Phi(y_2) + m_2. \quad \square$$

We do not need completeness. In fact $(i)$ and $(iii)$ together imply completeness.

[2] I omitted a proof here, try it as an exercise: If $a \succeq b$ and $a \sim c$, then $c \succeq b$.

# Question 6

We have the following three problems:

i.
$$\max_{x,z} \; \tilde{u}(x,z) \quad s.t. \; p \cdot x + \alpha z \leq w$$

ii.
$$\tilde{u}(x,z) = \max_{y} \; u(x,y) \quad s.t. \; q_0 \cdot y \leq z$$

iii.
$$\max_{x,y} \; u(x,y) \quad s.t. \; p \cdot x + \alpha q_0 \cdot y \leq w$$

# Question 6

"if part"
Let $(x^*, z^*)$ solves (i). This means that $\tilde{u}(x^*, z^*)$ is well defined.Therefore, from (ii), there exists $y'$ such that $\tilde{u}(x^*, z^*) = u(x^*, y')$ and $q_0 \cdot y' \leq z^*$.We know $y$ is *desirable*: $q_0 \cdot y' = z^*$.
Let $(x, y)$ be an arbitrary consumption bundle with $p \cdot x + \alpha q_0 \cdot y \leq w$[3].

$$u(x, y) \leq \tilde{u}(x, q_0 y) \leq \tilde{u}(x^*, z^*) = u(x^*, y')$$

□

[3]It is feasible for the whole consumer problem.

# Question 6

"only if part"
Let $(x^*, y') \in \arg\max_{x,y} \ u(x, y) \quad s.t. \ p \cdot x + \alpha q_0 \cdot y \leq w$.We have to show that $(x^*, q_0 \cdot y')$ solves (i) with $z^* = q_0 \cdot y'$.

$\forall$ $y$ such that $q_0 \cdot y \leq q_0 \cdot y'$ : $p \cdot x^* + \alpha q_0 \cdot y \leq p \cdot x^* + \alpha q_0 \cdot y' \leq w$[4]

This means that $u(x^*, y) \leq u(x^*, y')$ from optimality in consumer problem.Therefore $\tilde{u}(x^*, q_0 \cdot y') = u(x^*, y')$ from (ii). Let $(x, z)$ be any arbitrary bundle with $p \cdot x + \alpha z \leq w$. This means that any $y$ that solves (i) satisfies
$p \cdot x + \alpha q_0 \cdot y \leq p \cdot x + \alpha z \leq w$. So from (i) we have:
$\tilde{u}(x, z) \leq u(x^*, y') = \tilde{u}(x^*, q_0 \cdot y')$ □

[4] Again, this is just the feasibility of $(x^*, y)$.